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On compact composition operators acting between Bergman spaces

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Abstract

In this note we consider the compact composition operator acting different weighted Bergman spaces of the unit ball of $\mathbb{C}^N$. We will give an estimate for the essential norm of the composition operator. As a corollary, we can characterize the compactness of this operator in terms of the boundary behavior of the symbol.

1 Introduction

For a fixed integer $N>1$, let $\mathbb{C}^N$ denote the complex N-dimensional Euclidean space and B denote the open unit ball of $\mathbb{C}^N$. For each p, $0<p<\infty$ and $\alpha>-1$, the weighted Bergman space $A_\alpha^p(B)$ is the space of all holomorphic functions f on B for which

$$\|f\|_\alpha^p=\int_B|f(z)|^p(1-|z|^2)^\alpha dV(z)<\infty.$$

Here dV denotes the normalized Lebesgue volume measure on B. When $1\leq p<\infty$ the space $A_\alpha^p(B)$ is a Banach space. In particular, the space $A_\alpha^2(B)$ is a functional Hilbert space with inner product

$$\langle f,g\rangle_\alpha=\int_B f(z)\overline{g(z)}(1-|z|^2)^\alpha dV(z).$$

Since each point evaluation is a bounded linear functional, $A_\alpha^2(B)$ has the reproducing kernel function which is given by

$$K_w^\alpha(z)=\frac{c_\alpha}{(1-\langle z,w\rangle)^{\alpha+N+1}},$$

where $c_\alpha=1/\int_B(1-|z|^2)^\alpha dV(z)$.

Let φ be a holomorphic self-map of B, that is

$$\varphi=(\varphi_1,\ldots,\varphi_N)\,:\,B\to B,$$

where each φ_j is a holomorphic function on B. Then φ induces the composition operator C_φ, defined on the space of all holomorphic functions on B by

$$C_\varphi f = f \circ \varphi.$$

Many authors have studied these operators on various holomorphic function spaces. For these studies, see the monograph [3]. In this note, we discuss this operator on $A^p_\alpha(B)$. In the one variable case, Littlewood's subordination principle shows that every holomorphic function φ on the unit disk $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$ induces the bounded composition operator C_φ on the weighted Bergman space $A^p_\alpha(\mathbb{D})$. Thus the concern with the compactness of C_φ had been growing since the end of the last century. In 1986 B.D. MacCluer and J.H. Shapiro [5] gave a characterization for the symbol φ which induces the compact composition operator on $A^p_\alpha(\mathbb{D})$ as follows.

Theorem 1. *Let $0 < p < \infty$, $\alpha > -1$ and φ be a holomorphic function on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then the composition operator C_φ is the compact operator on $A^p_\alpha(\mathbb{D})$ if and only if φ satisfies the condition*

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0. \tag{1}$$

By Julia-Carathéodory's theorem we see that the above condition (1) is equivalent to φ has no finite angular derivative at any point of the boundary of $\mathbb{D}$.

The several variables (unit ball) case have some difficulties on the property of the composition operator C_φ. For instance, there is a holomorphic self-map of B such that the composition operator is not bounded on $A^p_\alpha(B)$. It is easy to construct the example. For the sake of the simplicity, we consider the case $N = 2$ and $p = 2$. We put $\varphi(z) = (2z_1z_2, 0)$ and consider the test function $f_k(z)$ defined by

$$f_k(z) = \sqrt{\frac{\Gamma(k+\alpha+3)}{k!\Gamma(\alpha+3)}}\, z_1^k \qquad (z = (z_1, z_2) \in B),$$

for $k \geq 1$ positive integer. Then $\{f_k\}$ is bounded in $A^2_\alpha(B)$ with $\sup_{k\geq 1} \|f_k\|_\alpha = 1$ and

$$f_k(\varphi(z)) = \sqrt{\frac{\Gamma(k+\alpha+3)}{k!\Gamma(\alpha+3)}}\, 2^k z_1^k z_2^k.$$

This implies that $\|C_\varphi f_k\|_\alpha \sim k^{\frac{1}{2}}$, and so C_φ is not bounded on $A^2_\alpha(B)$. When we study on the compact composition operator in the case $N \geq 2$, hence, we will need some assumptions which verify the boundedness of C_φ. For an univalent holomorphic self-map of B, the following sufficient condition for the boundedness of C_φ is known.

Theorem 2. *Suppose that an univalent holomorphic self-map of B which satisfies*

$$\sup_{z\in B} \frac{\|\varphi'(z)\|^2}{|J_\varphi(z)|^2} < \infty. \tag{2}$$

Then C_φ is bounded on $A^p_\alpha(B)$.

However it is also known that the condition (2) is not a necessary condition for the boundedness of C_φ. See [3, p.247]. Hence many authors have tried to characterize the compactness of C_φ on $A^p_\alpha(B)$ under some assumptions.

2 Well-Known Results

In [5], B.D. MacCluer and J.H. Shapiro also gave the following characterization.

Theorem 3. *Suppose that φ is an univalent holomorphic self-map of B which satisfy the condition* (2) *in Theorem 2. Then C_φ is compact on $A^p_\alpha(B)$ if and only if φ has no finite angular derivative at any point of the boundary of B.*

This result is the higher dimensional case of Theorem 1.

D.D. Clahane [2] proved the following result.

Theorem 4. *Let $p > 0$ and $\alpha \geq 0$. Suppose that φ is a holomorphic self-map of B such that C_φ is bounded on $A^p_\alpha(B)$ and φ satisfies the following condition*

$$\lim_{|z|\to 1^-} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2} \|\varphi'(z)\|^2 = 0.$$

Then C_φ is compact on $A^p_\beta(B)$ for all $\beta \geq \alpha$.

Clahane's result does not require the assumption φ is univalent but the relation between the compactness of C_φ and the boundary behavior of φ became unclear. Furthermore the spaces $A^p_\alpha(B)$ is restricted to the case $\alpha \geq 0$.

Recently, K. Zhu [8] have given the following characterization.

Theorem 5. *Let $p > 0$ and $\alpha > -1$. Suppose that C_φ is bounded on $A^q_\beta(B)$ for some $q > 0$ and $-1 < \beta < \alpha$. Then C_φ is compact on $A^p_\alpha(B)$ if and only if φ satisfies*

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

Note that Julia-Carathéodory's theorem for the unit ball case implies that the above condition is equivalent to φ has no finite angular derivative at any point of the boundary of B. Zhu's result does not also require the univalency of φ. Since he gave the characterization for the compactness of C_φ in terms of the angular derivative condition, we can consider that this result is the improved version of Theorem 3 or the higher dimensional case of Theorem 1.

In Theorem 3, Theorem 4 or Theorem 5, their results need some hypotheses on the symbol φ. The reason to need these assumptions on φ seems to be a technical request in their proof. Since every holomorphic self-map φ of B does not induce the bounded composition operator on $A^p_\alpha(B)$, the assumption that C_φ is bounded on $A^p_\alpha(B)$ is very natural condition for the unit ball case.

3 Main Result

Under the condition C_φ is bounded on $A^p_\alpha(B)$, we will consider the compactness problem. Recall that the essential norm of the bounded operator on Banach spaces. Let X and Y be Banach spaces. For a bounded operator $T : X \to Y$, the essential norm $\|T\|_{e,X\to Y}$ of T is defined to be the distance from T to the set of compact operators, namely $\|T\|_{e,X\to Y}$ is defined by

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\| : K \text{ is compact from } X \text{ to } Y\}.$$

Here $\|\ \|$ denotes the usual operator norm. By this definition, we see that $T : X \to Y$ is a compact operator if and only if $\|T\|_{e,X\to Y} = 0$. Thus the essential norm is closely related to the compactness problem of concrete operators. In Theorem 3, Theorem 4 and Theorem 5, they have not mentioned the essential norm of C_φ. In this note we give an estimate for the essential norm of $C_\varphi : A^2_\alpha(B) \to A^2_\beta(B)$ $(-1 < \alpha \le \beta)$.

Theorem 6. *Let $\alpha > -1$ and $\beta \ge \alpha$. Suppose that φ is a holomorphic self-map of B such that $C_\varphi : A^2_\alpha(B) \to A^2_\beta(B)$ is bounded. Then the essential norm of C_φ is comparable to*

$$\limsup_{|z|\to 1^-} \frac{(1-|z|^2)^{\beta+N+1}}{(1-|\varphi(z)|^2)^{\alpha+N+1}}.$$

So $C_\varphi : A^2_\alpha(B) \to A^2_\beta(B)$ is compact if and only if φ satisfies

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)^{\beta+N+1}}{(1-|\varphi(z)|^2)^{\alpha+N+1}} = 0.$$

In the previous our works [6, 7], we have the following characterization for the boundedness and compactness of $C_\varphi : A^p_\alpha(B) \to A^p_\beta(B)$.

Theorem 7. *Let $0 < p < \infty$ and $-1 < \alpha, \beta < \infty$. Suppose that φ is a holomorphic self-map of B. Then the following conditions are equivalent.*

(a) *$C_\varphi : A^p_\alpha(B) \to A^p_\beta(B)$ is a bounded operator,*

(b) *φ satisfies the condition*

$$\sup_{z\in B} \int_B \left\{ \frac{1-|z|^2}{|1-\langle \varphi(w), z\rangle|^2} \right\}^{\alpha+N+1} dV_\beta(w) < \infty.$$

Here dV_β denotes the weighted measure $dV_\beta(w) = (1-|w|^2)^\beta dV(w)$. Moreover,

(c) *$C_\varphi : A^p_\alpha(B) \to A^p_\beta(B)$ is a compact operator,*

(d) *φ satisfies the condition*

$$\sup_{|z|\to 1^-} \int_B \left\{ \frac{1-|z|^2}{|1-\langle \varphi(w), z\rangle|^2} \right\}^{\alpha+N+1} dV_\beta(w) = 0.$$

This theorem shows the following result.

Corollary 1. *The boundedness and compactness of the composition operator* $C_\varphi : A^p_\alpha(B) \to A^p_\beta(B)$ *are independent of the exponent* p.

Combining Theorem 6 with Corollary 1, we have the following characterization.

Corollary 2. *Let* $0 < p < \infty$ *and* $-1 < \alpha \leq \beta$. *Suppose that* φ *is a holomorphic self-map of* B *which induces the bounded composition operator* $C_\varphi : A^p_\alpha(B) \to A^p_\beta(B)$. *Then* $C_\varphi : A^p_\alpha(B) \to A^p_\beta(B)$ *is compact if and only if*

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)^{\beta+N+1}}{(1-|\varphi(z)|^2)^{\alpha+N+1}} = 0.$$

According to the result due to J.A. Cima and P.R. Mercer [1], every holomorphic self-map φ of B induces the bounded composition operator $C_\varphi : A^p_\alpha(B) \to A^p_{\alpha+N-1}(B)$. Hence it would be very interesting to know the compactness criteria for this situation. Indeed, H. Koo has proposed the following problem in [4].

Characterize the compactness of the composition operator

$$C_\varphi : A^p_\alpha(B) \to A^p_{\alpha+N-1}(B).$$

Since we see that $\alpha + N - 1 > \alpha$ for $\alpha > -1$, this situation suits the assumption in Theorem 6. Thus we can give an answer to Koo's question as follows.

Corollary 3. *Let* $\alpha > -1$, $0 < p < \infty$ *and* φ *be a holomorphic self-map of* B. *Then* $C_\varphi : A^p_\alpha(B) \to A^p_{\alpha+N-1}(B)$ *is compact if and only if* φ *satisfies*

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)^{\alpha+2N}}{(1-|\varphi(z)|^2)^{\alpha+N+1}} = 0.$$

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